These are all of the symmetries, but what about the physical symmetries, i.e., the ones we could do if the triangle were stuck in the plane? A careful observation shows that we only lose the reflection, so the surviving elements of D_3 are

$$e, r, r^2$$

the rotations.

We call this group $\mathbb{Z}_3 = \langle r \mid r^3 = e \rangle$ (sometimes called the <u>cyclic group of 3 elements</u>) [more on this notation later].

Full Symmetry Group $= D_3 = \langle r, s \mid r^3 = s^2 = e, rs = sr^{-1} \rangle$

Physical Symmetry Group $= \mathbb{Z}_3 = \langle r \mid r^3 = e \rangle$

For a general regular n-gon:

(n-gon = polygon with n sides)

- Symmetry group $= D_n = \langle r, s \mid r^n = e, s^2 = e, sr = r^{-1}s \rangle$

In D_n, $r^{-1} = r^{n-1}$ since

$$r \cdot r^{-1} = e = r^n = r \cdot r^{n-1}$$

and

$$r^{-1} \cdot r = e = r^n = r^{n-1} \cdot r$$

<u>**Warning**</u> $\frac{1}{r}$ does not make sense in a group, necessarily. r^{-1} is what we use.

We can also see, by the same reasoning, that since $s^2 = e \Rightarrow s^{-1} = s$.

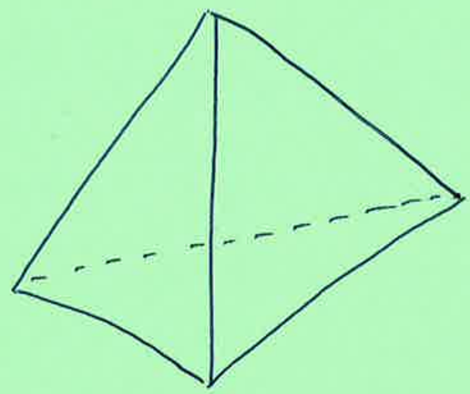
- Physical symmetry subgroup:

<u>only rotations</u>

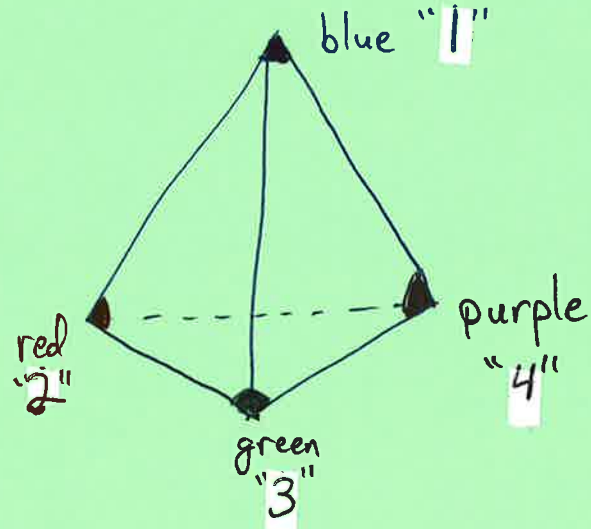
$$\mathbb{Z}_n = \langle r \mid r^n = e \rangle$$

Symmetries of the Regular Tetrahedron

Let's choose our standard position for the tetrahedron to be:



We can choose to keep track of where faces go or where vertices (corners) go. Since it's easier to color corners on paper, we will follow the corners in the notes (we did faces in class). Color the corners as:



We can keep track of the symmetries as follows:

1) Decide where the blue corner goes:
4 choices

2) Decide where the red corner goes:
3 choices

3) Decide where the purple corner goes:

Physically, this has already been decided, so there is only 1 choice!

However, if we allow ourselves to turn the tetrahedron inside out (by a reflection through some plane), we still have 2 choices!

General symmetry subgroup

any reordering of 1, 2, 3, 4 (called a permutation)

$S_4 = \{\text{all permutations of } 1,2,3,4\}$

Physical Symmetry Subgroup

This group has half the number of symmetries as the general symmetry group, S_4. The subgroup is slightly tricky to nail down. The name of the group is the <u>alternating group</u>

$$A_4 = \{\text{even permutations of } 1, 2, 3, 4\}$$

A permutation is <u>even</u> if it a product of an even number of transpositions, i.e., cycles of the form $(a\ b)$

The 12 cycles in A_4 are:

e	$(1\ 2\ 3)$	$(1\ 3\ 4)$
$(1\ 2)(3\ 4)$	$(1\ 3\ 2)$	$(1\ 4\ 3)$
$(1\ 3)(2\ 4)$	$(1\ 2\ 4)$	$(2\ 3\ 4)$
$(1\ 4)(2\ 3)$	$(1\ 4\ 2)$	$(2\ 4\ 3)$

We can factor a cycle $(a\ b\ c)$ as

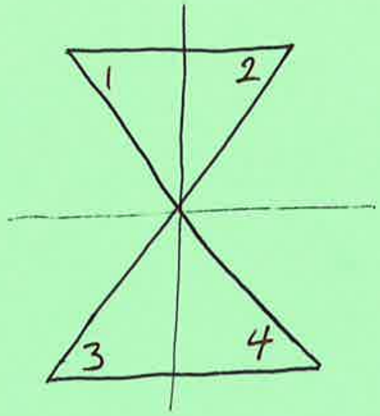
$$(a\ b\ c) = (a\ c)(c\ b)$$

to see that all of the 3-cycles listed are, in fact, even permutations!

An example of a permutation <u>not</u> in A_4 is $(2\ 4)$, since it is not physically possible to switch only corners 2 & 4.

<u>Exercise</u>: Can you get all of the remaining permutations in S_4 by multiplying elements of A_4 by $(2\ 4)$? Why do you think this is? What kind of symmetry is $(2\ 4)$?

Example: Find the symmetries of



Sol: There are 3 things:

- rotation by π

$$r = (1\ 4)(2\ 3)$$

- reflection about the vertical line

$$s_v = (1\ 2)(3\ 4)$$

- reflection about the horizontal line

$$s_h = (1\ 3)(2\ 4)$$

Notice that

$$r s_v = (1\ 4)(2\ 3)(1\ 2)(3\ 4) = (1\ 3)(2\ 4) = s_h$$

and

$$s_v r = (1\ 2)(3\ 4)(1\ 4)(2\ 3) = (1\ 3)(2\ 4) = s_h$$

So, the symmetry group is generated by r & s_v

$$r^2 = (1\ 4)(2\ 3)(1\ 4)(2\ 3)$$
$$= (1)(2)(3)(4) = e$$
$$s_v^2 = (1\ 2)(3\ 4)(1\ 2)(3\ 4)$$
$$= (1)(2)(3)(4) = e$$

Thus, the symmetry group is

$$\langle r, s_v \mid r^2 = e, s_v^2 = e, r s_v = s_v r \rangle$$

This is known as the Klein Four Group and is sometimes written V or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.